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The Paradox of Radiation from a Uniformly Accelerated Point Electron and a Consistent Physical Framework for its Resolution†

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Abstract

We study the problem of an electron in an external, uniform electric field. The problem is analyzed within the framework of a new theory of Maxwell electrodynamics in which the problems and paradoxes of infinite self-energies are omitted by considering only the interactions between charged particles as the basic building blocks of the theory. For this case of uniform acceleration, the manifest consistency and physical transparency of this new electrodynamic formalism allows a simple and physically clear interpretation to be ascribed to the 'acceleration energy', and the role it plays in acting as the source of the energy which is radiated from the accelerated charge. The relationship of the acceleration energy to the internal energy of the electron is clarified in terms of the 'total coupled radiation field' of the system. At the same time the new formulation of electrodynamics is shown to represent a completely consistent theory of a classical electron in an external field, which in many aspects is a superior alternative to either Maxwell–Lorentz theory or Wheeler–Feynman theory.

1. *Introduction*

The study of the motion of a point electron in an external field which causes it to undergo constant acceleration is a time-honored problem which over the past sixty years has received the attention of many prominent physicists (Schott, 1915; Milner, 1921; Drukey, 1949; Bondi & Gold, 1955; Rohrlich & Fulton, 1960). When the problem is analyzed from the point of view of standard Maxwell–Lorentz electrodynamics, one finds that the Lorentz–Dirac equation implies that the radiation reaction force vanishes for constant acceleration. However, since the rate of loss of radiation energy is not zero for an accelerating change, an apparent paradox ensues. This is because, if the radiation reaction force (proportional to the time derivative of the acceleration) vanishes, then the rate of work done by the external field will equal the time rate of increase of the particle's kinetic energy. Because of this exact balance of energies, it cannot be the external force which supplies the necessary radiation energy. Superficially,

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this represents an apparent violation of conservation of energy. Several analyses of this problem† have shown that if the Lorentz–Dirac equation is valid, then the required source of radiation energy is in the 'acceleration energy' of the particle. For non-relativistic motion, this is given by

$$Q = \left(\frac{2e^2}{3xc^3}\right) \mathbf{V} \circ \mathring{\mathbf{V}} \tag{1.1}$$

where e is the electron charge, and $\mathbf{V}$ is the electron's classical velocity.

However, the physical interpretation of this 'acceleration energy' (or 'Schott energy', as it is sometimes called) is obscure. Apparently, it represents an energy associated with the charged-point electron, which does not contribute to the renormalization of the electronic mass, nor does it represent a field energy which manifests itself in terms of long-range fields at infinity. Nevertheless, this energy can be shown to act as the source of the radiation energy, in the constant acceleration case, and is thus interpreted as being a component (albeit a physically obscure one) of the total internal energy of the electron.

The purpose of this paper is to re-analyze this problem within the framework of a newly developed theory of electrodynamics (Leiter, 1969), based on the concept that it is the mutual electromagnetic interactions between charged particles (and not the charged particles themselves) which are the basic 'elementary' building blocks of physical events in the electromagnetic domain. Recently, a theoretical energy paradox in classical electron theory (Leiter, 1970) was resolved within this new electromagnetic framework. Because of this success, it was felt that further applications were in order. With regard to the present problem under consideration, the manifest consistency and physical transparency of the new formalism allows a simple and physically clear meaning to be ascribed to the acceleration energy and its role in producing the radiation from a uniformly accelerated charged particle. The relationship of this acceleration energy to the internal energy of the electron is clarified, while at the same time the new formulation of electrodynamics is shown to represent a superior theory of classical point charges in external fields, than that of conventional Maxwell–Lorentz theory.

2. *Classical Elementary Measurement Electrodynamics in the External Field Approximation*

In a recent publication (Leiter, 1970) it was shown that a new formulation of classical electrodynamics could be developed on the basis of a new physical paradigm of the process of measurement. The new paradigm states that: *in a physical event, it is the mutual measurement interaction between the 'observer charge' and the 'observed charge' which is the basic*

† See, for example, Rohrlich, F. & Fulton, T. (1960). *Annals of Physics*, **9**, 449.

building of any theory attempting to describe that physical event.† Within the context of a relativistic theory of point charges, this new theory of electrodynamics implies that the charged particles [associated with $J_\mu^{(K)}(\mathbf{x},t)$; $K=1, 2,\ldots, N$] and their electromagnetic fields [associated with $A_\mu^{(K)}(\mathbf{x},t)$; $K=1, 2,\ldots, N$] are not elementary in themselves but are merely interdependent degrees of freedom in a scalar elementary measurement field $J_\mu^{(K)}A^{\mu(J)}$; $K\neq J=1,\ldots, N$. Because this implies that the scalar elementary measurement field $J_\mu^{(K)}A^{\mu(J)}$ is more fundamental that either $J_\mu^{(K)}$ or $A_\mu^{(J)}$, then in this theory Maxwell's equations (with the proper choice of Green function) are interpreted to be a set of covariant *identities* which give a prescription for converting particle currents $J_\mu^{(K)}$ into their associated electromagnetic fields $A_\mu^{(K)}$. Because all the electromagnetic fields $A_\mu^{(K)}$ are required to be *directly* connected to their associated currents $J_\mu^{(K)}$, through their associated Maxwell identities, then no free uncoupled electromagnetic fields will exist in the theory. This means that the phenomena of radiation will occur as the by-product of the propagation of mutual electromagnetic measurement interactions between charged particles, and will not be independent of the detector. Also, on the basis of the paradigm which underlies this theory, all self-measurement fields $J_\mu^{(K)}A^{\mu(K)}$; $K=1, 2,\ldots, N$ are excluded *a priori* as being unphysical. Hence the problem of infinite self-energies is excluded in the basic formulation of the theory, and never arises again in calculation. In addition, mass renormalization is unnecessary in this theory, instead, the mass parameters which appear in this theory are the empirical quantities which are determined by experiment. Within the context of this new theory, it has been shown that an action principle for the relativistic elementary measurement of N classical point charges, and their associated electromagnetic fields, can be constructed. Making the action stationary with respect to the variation of the interacting particle and field degrees of freedom yields the equations of motion of the measurement. With the proper choice of Green function for the associated N-column Maxwell tautologies, the equations of motion have the same form as those of the renormalized Lorentz–Dirac equations of Maxwell–Lorentz electrodynamics. However, the absence of self-interactions implies that no

† At this point, the author wishes to make a distinction between the work presented here, and that of Sachs, M. and Schwebel, S. (1961). *Nuovo cimento*, **21**, 197, Suppl. No. 2. Even though the present theory is a classical one, a similar basic paradigm has been used previously by the above authors in a wave-mechanical theory. However, the similiarity between the work presented in this paper, and that of Sachs and Schwebel lies only in the fact that a similar basic paradigm is used. In all other respects the present work is different in its structure and interpretation. Specifically, an important basic difference between the author's work and the paper by Sachs and Schwebel lies in the fact that in the latter paper time-symmetric potentials are used, but without invoking any complete absorber condition (*à la* Wheeler–Feynman). Hence, that formalism is unable to make an agreement with the results of conventional classical electrodynamics in a classical correspondence limit. On the other hand, the author's work presented in this paper, since it contains the Lorentz–Dirac equation, is in agreement with the well-known predictions of conventional electrodynamics, with retardation and radiation reaction properly accounted for.

'infinite' mass renormalization is required in these equations of motion. Hence a consistent set of energy-momentum and angular-momentum conservation laws follows directly from the associated energy-momentum tensor of the theory, which are free of the difficulties associated with the infinite mass renormalization problem of conventional Maxwell–Lorentz theory.

Within this framework, the case of a single particle in an 'external field' corresponds to the case† where a single charge is artificially isolated from the others, and the approximation is made that the effect of the 'aggregate' of other charges, on the single charge, is negligibly affected by the reaction of the single particle back to the 'aggregate' (i.e. the 'external' field's source). In this case the equation of motion of the single electron, labeled by $x_\mu^{(1)}(t)$, is given by

$$\frac{dp_\mu^{(1)}}{dt} = \left(\frac{q^{(1)}}{c}\right)\frac{dx_\nu^{(1)}}{dt} F_{\nu\mu(\text{ext.})}\Big|^{\text{traj.}} \tag{2.1}$$

where the associated electromagnetic potentials of the 'external' field and the single electron are given by

$$A_{\mu(\text{ext.})}^{(x)} = \int dx'^4\, D_{(+)}(x-x')J_\mu(x')_{\text{ext.}} + \int dx'^4\, D_{(-)}(x-x')\,[J_\mu^{(1)}(x') + J_\mu(x')_{\text{ext.}}] \tag{2.2}$$

$$A_\mu^{(1)}(x) = \int dx'^4\, D_{(+)}(x-x')J_\mu^{(1)}(x') + \int dx'^4\, D_{(-)}(x-x')\,[J_\mu^{(1)}(x') + J_\mu(x')_{\text{ext.}}] \tag{2.3}$$

respectively. In the above, $D_{(+)} = (D_{\text{ret.}} \pm D_{\text{adv.}})/2$, where $\Box D_{\substack{\text{ret.}\\ \text{adv.}}}(x-x') = \delta^4(x-x')$.

Inserting (2.2) into (2.1), we have the conventional Lorentz–Dirac equation for the single electron in the external field as

$$\frac{dp_\mu(1)}{dt} = \left(\frac{q(1)}{c}\right)\frac{dx_\nu(1)}{dt}\,[F_{\nu\mu(\text{ret.})\text{ext.}} + F_{\nu\mu(-)}^{(1)}]\,\big|^{\text{traj.}} \tag{2.4}$$

In going from (2.1) to (2.4) we see that the electron sees some of the retarded external Lorentz force and the Abraham radiation-reaction force as an interference between the 'time-symmetric' Lorentz force and the 'total coupled radiation' Lorentz force of the theory. This occurs because of the presence of the 'total coupled radiation field' in this new formulation of electrodynamics.‡ We will see shortly that it is the interaction of the

† See Leiter, D. (1969). *Annals of Physics*, **51**, No. 3, pp. 568–570.

‡ This was shown by Leiter (1969). There it was demonstrated that the theory contained a 'total coupled radiation field' directly coupled to the currents of the charged particle of the system, but obeying a homogeneous wave equation. It can be shown that this total coupled radiation field can be written as a free wave packet whose Fourier coefficient is *directly* connected to the total current of the system. Interference between this total

electron's retarded fields with the total coupled radiation field of the theory which can account for the 'acceleration energy' of the electron. Before doing this, it should be noted that the structure of this theory has the advantage of giving a physical interpretation to the phenomena of radiation reaction within the context of a Lagrangian formalism. The lack of infinite renormalization problems and the lack of any cosmological assumption about 'complete absorption' (Wheeler & Feynman, 1945, 1949), implies that this theory represents a completely consistent structure for the description of an electron in an external field. Hence there are no mathematical difficulties or infinite subtraction processes involved in making calculations with the equations of motion and the conservation laws. From Leiter (1970), the associated conservation laws which accompany equation (2.4) are (for the energy conservation)

$$\frac{d}{dt}\left[\rho_0(1) + \int_v dx^3(\mathbf{E}_{(1)}\cdot\mathbf{E}_{\text{ext.}} + \mathbf{B}_{(1)}\cdot\mathbf{B}_{\text{ext.}})\right] + \int_v dx^3(\mathbf{J}_{\text{ext.}}\cdot\mathbf{E}_{(1)}) = -\oint_s \mathbf{ds}\cdot(\mathbf{E}_{(1)}\times\mathbf{B}_{\text{ext.}} + \mathbf{E}_{\text{ext.}}\times\mathbf{B}_{(1)})\,c \tag{2.5}$$

when (2.2) and (2.3) are inserted into (2.5), and the assumption is made that the external field is a purely electric one (which is uniform and time independent within the spatial volume of interest), then (2.5) becomes†

$$\frac{d}{dt}\left[\rho_0(1) + \int dx^3(\mathbf{E}^{(1)}_{\text{ret.}}\cdot\mathbf{E}^{(1)}_{(-)} + \mathbf{B}^{(1)}_{\text{ret.}}\,\mathbf{B}^{(1)}_{(-)}) + q^{(1)}\phi_{\text{ext.(ret.)}}|^{\text{traj.}}\right] = \left(-\oint \mathbf{ds}\cdot(\mathbf{E}^{(1)}_{\text{ret.}}\times\mathbf{B}^{(1)}_{\text{ret.}})\right) \tag{2.6}$$

while the equation of motion (2.4) becomes (now assuming that $v \ll c$)

$$\begin{aligned}\frac{d\mathbf{p}(1)}{dt} &= q\mathbf{E}_{\text{ext.(ret.)}}|^{\text{traj.}} + \frac{2q^2}{3c^3}(\ddot{\mathbf{V}}(1)) \\ &= q\mathbf{E}_{\text{ext.(ret.)}} \qquad (\text{if } \ddot{\mathbf{V}} = 0)\end{aligned} \tag{2.7}$$

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coupled radiation field, and that of the time-symmetric, mutual interaction fields, was shown to yield the conventional retarded electromagnetic fields and the radiation reaction field, if more than two charged particles existed in the model universe. That this radiation reaction field was not due to direct self-interaction of a particle on itself, was shown by the fact that if the model universe contained only one charged particle, then the radiation reaction field action on this particle was zero. In passing, we note that the existence of a total radiation field is another basic difference in the author's theory and that of the work discussed in the footnote on p. 389.

† In going from equation (2.5) to equation (2.6), we use the fact that the external electric field is static and uniform in a finite volume, and must go to zero at spatial infinity, faster than $0(1/r)$, because of the absence of 'externally' induced radiation fields. In addition we use the fact that the wave-zone fields have the property that

$$(\mathbf{E}^{(1)}_{\text{ret.}}\times\mathbf{E}^{(1)}_{\text{adv.}} + \mathbf{E}^{(1)}_{\text{adv.}}\times E^{(1)}_{\text{ret.}}) = 0$$

Using the Lienard–Wiechert potentials in (2.6), we have that

$$\frac{d}{dt}\left[\rho_0(1)+\int dx^3(\mathbf{E}^{(1)}_{\text{ret.}}\cdot\mathbf{E}^{(1)}_{(-)}+\mathbf{B}^{(1)}_{\text{ret.}}\cdot\mathbf{B}^{(1)}_{(-)})\right]=q\mathbf{V}(1)\cdot\mathbf{E}_{\text{ext.(ret.)}} -(2q^2/3c^3)(a(1))^2 \tag{2.8}$$

The conservation law (2.8) which accompanies the equation of motion (2.7), implies that an additional energy quantity is associated with the moving electron, besides its kinetic energy. It is the sum of these two energies whose negative time rate of change of energy feeds the radiation power lost to infinity. In this theory, this extra energy represents the interaction of the electron's retarded fields with the total coupled radiation field of the system. In this special case, where the external electric field is considered to be uniform and time independent in the volume of interest, the total coupled radiation field is

$$\begin{gathered} A^{\mu}_{\text{ext.}}(-)+A^{\mu}_{(1)}(-)\equiv A^{\mu}_{\text{rad.}}; \qquad A^{\mu}_{\text{ext.}}(-)=0 \\ \mathbf{E}_{\text{rad.}}=\mathbf{E}_1(-); \qquad \mathbf{B}_{\text{rad.}}=\mathbf{B}_1(1) \end{gathered} \tag{2.9}$$

The direct identification of this new electromagnetic interaction term, with the acceleration energy can be seen from the equation of motion (2.7), as

$$\frac{d\rho_0^{(1)}}{dt}=q\mathbf{V}(1)\cdot\mathbf{E}_{\text{ext.(ret.)}}\big|^{\text{traj.}}+\frac{2q^2}{3c^3}\left[\frac{d}{dt}(\mathbf{V}(1)\cdot\mathring{\mathbf{V}}(1))-\mathring{\mathbf{V}}(1)\cdot\mathring{\mathbf{V}}(1)\right] \tag{2.10}$$

Hence we see that the acceleration energy of the electron ($q=e$) is from (2.8) and (2.10)

$$Q=\left(\frac{2e^2}{3c^3}\right)\mathbf{V}(1)\cdot\mathring{\mathbf{V}}(1)=-\int_v dx^3(\mathbf{E}_{(\text{ret.})}\cdot\mathbf{E}_1(-)+\mathbf{B}_{1(\text{ret.})}\cdot\mathbf{B}_1(-)) \tag{2.11}$$

is thereby given a physical interpretation within the Classical Elementary Measurement Electrodynamic framework. Since there is no renormalization in this theory, then it is obvious that Q does not contribute to any change in the electron's mass. In addition, since Q involves the $\mathbf{E}_1(-)$ and $\mathbf{B}_1(-)$ fields, it does not manifest itself at infinity, in terms of long-range fields, as expected. However, equations (2.7)–(2.11) show directly that it is the depletion of this Q energy which accounts for the source of the radiation energy. Since (2.8) is true for relativistic velocities, then the role of the acceleration energy (2.11) remains the same even if $v\sim c$ [even though the more complicated equation (2.4) must be used in place of (2.7)]. This formalism can be easily extended to the case of external electric and magnetic fields (either uniform or time-dependent) in which case extra terms are introduced into the equations of motion and conservation laws. However, the role of the acceleration energy as related to the interaction of the electron's retarded fields with the total coupled radiation field of the theory (now generalized to include the effect of the time dependence of the external fields, if any) remains essentially the same.

3. *Conclusions*

We have shown that within the framework of Classical Elementary Measurement Electrodynamics, a completely consistent theory of the motion of an electron in an external field can be developed. The lack of infinite self-energies allows direct calculations to be carried out without mathematical difficulties, and eliminates the difficulties of mass renormalization in the associated conservation laws. A new feature of the theory is the presence of the 'total coupled radiation field' which accounts for the phenomena of radiation reaction and is able to give a simple physical meaning to the presence of the 'acceleration energy', as regards an electron in a uniform electric field. The acceleration energy (which acts as the source of the radiation energy sent to infinity, for uniform acceleration) is shown to be due to the interaction of the electron's retarded fields with the total coupled radiation fields of the system. This interaction can be thought of as being part of the electron's internal energy, in addition to its rest energy. This can occur because the Lorentz–Dirac equation is actually *more* compatible with Classical Elementary Measurement Electrodynamics, than it is with conventional Maxwell–Lorentz electrodynamics. This is because the Lorentz–Dirac equation (with the physical electron mass) arises in Maxwell–Lorentz theory only *after* the infinite mass renormalization process is carried out. In Classical Elementary Measurement Electrodynamics, the Lorentz–Dirac equation arises directly from the Lagrangian formalism and the structure of the associated Maxwell equations, *without* any infinite mass renormalization process ever occurring. Because of the mathematical inconsistencies in Maxwell–Lorentz theory of the electron, it is not surprising that the presence of the acceleration energy in the Lorentz–Dirac equation seems obscure and physically unclear. On the other hand, the complete consistency of the Classical Elementary Measurement Electrodynamic electron theory, and its compatibility with the Lorentz–Dirac equation (without infinite renormalizations), allows a direct physical interpretation of the acceleration energy to emerge. As far as the classical electrodynamics of electrons is concerned, it would seem that Classical Elementary Measurement Electrodynamics is a superior alternative to either Maxwell–Lorentz theory or Wheeler–Feynman theory.

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